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Concordant outranking with multiple criteria of ordinal significance

A contribution to robust multicriteria aid for decision

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Abstract In this paper we address the problem of aggregating outranking statements from multiple preference criteria of ordinal significance. The concept of ordinal concordance of a global outranking situation is defined and an operational test for its presence is developed. Finally, we propose a new kind of robustness analysis for global outranking statements integrating classical dominance, ordinal and classical majority concordance in a same ordinal valued logical framework.

Keywords: Multicriteria Decision Aid, Outranking Methods, Concordance Approach, Ordinal Significance of Criteria

Introduction

In multicriteria decision aid methodology, and more particularly in the outranking methods, the problem of aggregating preference statements along multiple points of view is commonly solved with the help of cardinal weights translating the importance the decision maker gives each point of view (Roy, 1985, 1991). However, determining the exact numerical values of these weights remains one of the most obvious practical difficulties in applying outranking methods (Roy and Mousseau, 1996). Similar problems arise in the MAUT methodology (see Zeleny, 1982), but we shall here concentrate essentially on the outranking methodology.

To cope with the difficulty of measuring the exact numerical importance of each point of view in a given decision problem, we extend in a first section the majority concordance principle – as implemented in the ELECTRE methods (Roy, 1985, 1991) – to the context where merely ordinal information concerning the relative weights of the criteria is available. Basic data and notation is introduced and the classical outranking concept is adapted to our purpose. The ordinal concordance principle is formally introduced and illustrated on a simple car selection problem.

In a second section, we address the theoretical foundation of our definition of ordinal concordance. In addition, an operational test for assessing the truthfulness of an ordinally concordant outranking statement is developed. The core approach involves the construction of a distributional dominance test similar in its design to the stochastic dominance approach.

In a third section we finally address the general robustness problem of valued outranking statements. Classical dominance, i.e. unanimous concordance, ordinal as well as simple majority concordance, are considered altogether in a common logical framework in order to achieve robust optimal choice recommendations. We rely in this approach on recent work of us on good choices methodology from ordinal valued outranking relations (see Bisdorff and Roubens, 2004).

1 The ordinal concordance principle

We start with setting up the necessary notation and definitions. Here we more or less follow the notation used in the French multicriteria decision aid community.

1.1 Basic data and notation

As starting point, we require a set A of potential decision actions. To assess binary outranking situations between these actions we consider a consistent family $F = \{g_1, \ldots, g_n\}$ of n preference criteria (Roy and Bouyssou, 1993, Chapter 2). The performance tableau gives us for each decisions action $a \in A$ its corresponding performance vector $g(a) = \big(g_1(a), ..., g_n(a)\big)$.

A first illustration – shown in Table 1 – concerns a simple car selection problem taken from Vincke (1992, pp. 61–62)). We consider here a set $A = \{m_1, \ldots, m_7\}$ of seven potential car models which are evaluated on four criteria: *Price*, *Comfort*, *Speed* and *Design*. In this supposedly consistent family of criteria, the *Price* criterion works in the negative direction of the numerical amounts. The evaluations on the qualitative criteria such as *Comfort*, *Speed* and *Design* are numerically coded as follows: 3 means *excellent* or *superior*, 2 means *average* or *ordinary*, 1 means *weak*.

In general, we may observe on each criterion $g_j \in F$ an indifference threshold $q_j \geqslant 0$, a strict preference threshold $p_j \geqslant q_j$ and a veto threshold $v_j > p_j$. (see Roy and Bouyssou, 1993, pp. 55–59). We suppose for instance that the decision-maker admits on the *Price* criterion an indifference threshold of 10, a preference

Table 1. Car selection problem: performance tableau

Cars	q_j	p_j	v_j	m_1	m_2	m_3	m_4	m_5	m_6	m_7	w
1: Price	10	50	150	300	270	250	210	200	180	150	5/15
2: Comfort	0	1	2	3	3	2	2	2	2	1	4/15
3: Speed	0	1	–	3	2	3	3	2	3	2	3/15
4: Design	0	1	–	3	3	3	1	3	2	2	3/15

Source:Vincke, Ph. 1992, pp. 61–62

threshold of 50 and a veto threshold of 150 units. On the *Comfort* criterion we express a veto against the global outranking of a weakly comfortable against a superior comfortable car.

To simplify the formal exposition, we consider in the sequel that all criteria support the decision maker's preferences along a positive direction. Let $\Delta_j(a,b) = g_j(a) - g_j(b)$ denote the difference between the performances of the decision actions a and b on criterion g_j. On each criterion $g_j \in F$, we denote "$a\,\mathrm{C}_j\,b$" the semiotic restriction of assertion "a is evaluated at least as good as b" to the individual criterion g_j.

Definition 1 (Local preference assessment).
$\forall a, b, \in A$, the level of credibility $r(a\,\mathrm{C}_j\,b)$ of assertion "$a\,\mathrm{C}_j\,b$" is defined as:

$$r(a\,\mathrm{C}_j\,b) = \begin{cases} 1 & \textit{if} \quad \Delta_j(a,b) \geqslant -q_j \\ \frac{p_j + \Delta_j(a,b)}{p_j - q_j} & \textit{if} \quad -p_j \leqslant \Delta_j(a,b) \leqslant -q_j \\ 0 & \textit{if} \quad \Delta_j(a,b) < -p_j. \end{cases} \tag{1}$$

The level of credibility $r(\overline{a\,\mathrm{C}_j\,b})$ associated with the truthfulness of the negation of the assertion "$a\,\mathrm{C}_j\,b$" is defined as follows:

$$r(\overline{a\,\mathrm{C}_j\,b}) = 1 - r(a\,\mathrm{C}_j\,b). \tag{2}$$

Following these definitions, we find in Table 1 that model m_6 is clearly evaluated at least as good as model m_2 on the *Price* criterion ($\Delta_1(m_6, m_2) = 90$ and $r(m_6\,\mathrm{S}_1\,m_2) = 1$) as well as on the *Speed* criterion ($\Delta_3(m_6, m_2) = 1$ and $r(m_6\,\mathrm{S}_3\,m_2) = 1$).

Inversely, model m_2 is also clearly evaluated at least as good as model m_6 on the *Comfort* criterion as well as on the *Design* criterion. Indeed $\Delta_2(m_2, m_6) = 2$ so that $r(m_2\,\mathrm{C}_2\,m_6) = 1$ and $\Delta_4(m_2, m_6) = 1$ gives $r(m_2\,\mathrm{S}_4\,m_6) = 1$.

A given performance tableau, if constructed as required by the corresponding decision aid methodology (see Roy, 1985), is warrant for the truthfulness of these "*local*", i.e. individual criterion based preferences of the decision maker. However, to assess global preference situations integrating all available criteria, we need to aggregate these local warrants by considering the relative importance the decision-maker attributes to each individual preference dimension with respect to his global preference system. For an individual preference dimension, this importance is captured by the relative *significance*, which takes the criterion function modelling the preferences along this dimension. Such a criterion function is working – in a Peircean sense[1] – as an *iconic sign* of the local outranking situation. The relative importance of the preference dimension modelled by the criterion is translated by the relative *significance* this particular icon takes in warranting the truthfulness of a global outranking statement[2].

1.2 The majority concordance principle

In the ELECTRE based methods, assessing the global outranking situation is addressed by evaluating if, yes or no, a *more or less significant* majority of criteria effectively agree on supporting a given global "at least as good" assertion and, whether no local veto is expressed against it (see Roy and Bouyssou, 1993; Bisdorff, 2002). This outranking approach for assessing aggregated preferences from multiple criteria was originally introduced by Roy (1968, 1991).

Definition 2 (Concordant outranking index).
Let $w = (w_1, ..., w_n)$ be a set of significance weights corresponding to the n criteria such that: $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$. For $a, b \in A$, let $a \,\mathrm{C}\, b$ denote the assertion that "a concordantly outranks *b". We denote $r_w(a \,\mathrm{C}\, b)$ the level of concordance of assertion $a \,\mathrm{C}\, b$ considering given significance weights w.*

$$r_w(a \,\mathrm{C}\, b) \;=\; \sum_{j=1}^{n} \big(w_j \cdot r(a \,\mathrm{C}_j\, b)\big). \tag{3}$$

Assertion "$a \,\mathrm{C}\, b$" is considered rather true than false*, as soon as the weighted sum of criterial significance in favour of the global outranking statement obtains a strict majority, i.e. the weighted sum of criterial significance is greater than 50%.*

Following the general definition of an outranking situation (see Roy, 1985), we shall combine the concordant outranking test with a non veto test.

Definition 3 (Global outranking index).
Let "$a \,\mathrm{S}\, b$" denote, as usual in the outranking methods, the assertion that "a (globally) outranks *b" and let "$a \,\mathrm{D}\, b$" denote the fact that there exists a veto expressed against assertion $a \,\mathrm{S}\, b$, i.e. $\exists\, g_j \in F$ such that $\Delta_j(a, b) \leqslant -v_j$. We denote*

[1] See C.S. Peirce, *Logic as Semiotic: The Theory of Signs*. In Buchler (ed.) (1955).

[2] The significance of a criterion function differs from the concept of importance of a criterion as arising in the MAUT context (Zeleny, 1982), where the criterion function is not an icon but a conventional symbolic construction, a utility measure. The importance of the individual preference dimension in the global utility measure is taken into account with specific importance weights based on utility substitution rates.

$r_w(a\,\mathrm{S}\,b)$ *the credibility index of assertion "$a\,\mathrm{S}\,b$" when considering significance weights w:*

$$r_w(a\,\mathrm{S}\,b) \;=\; \begin{cases} r_w(a\,\mathrm{C}\,b) & \textit{if} \quad (\overline{a\,\mathrm{D}\,b})\,; \\ 0 & \textit{otherwise}. \end{cases} \tag{4}$$

Credibility of the global outranking assertion is given by the concordant outranking level if there does not exist any veto expressed against it. Otherwise, the outranking credibility is put to zero[3].

To clearly show the truth-functional denotation implied by the credibility function r_w, we introduce some further notations. Let w represent a given set of significance weights.

Definition 4 (Truth denotation of the outranking index).
We denote $[\![a\,\mathrm{S}\,b]\!]_w$ the truth-functional evaluation of assertion "$a\,\mathrm{S}\,b$" based on the outranking index $r_w(a\,\mathrm{S}\,b)$ and taking its values in a three valued truth domain $L_3 = \{f_w, u, t_w\}$ where: f_w means rather false than true *considering importance weights w, t_w means* rather true than false *considering importance weights w and, -u means* logically undetermined.

$$[\![a\,\mathrm{S}\,b]\!]_w \;=\; \begin{cases} t_w & \textit{if} \quad (\,r_w(a\,\mathrm{S}\,b) > 0.5\,)\;; \\ f_w & \textit{if} \quad (\,r_w(a\,\mathrm{S}\,b) < 0.5\,)\;; \\ u & \textit{otherwise}. \end{cases} \tag{5}$$

In Definition 4, we do not follow the traditional ELECTRE methodology which introduces for this purpose the concept of confidence level s of the outranking index, a further rather delicate parameter to fix in practical applications. From recent theoretical results of ours we know indeed that there is no operational need to introduce – apart from the 50% cut as illustrated in Equation 5 above – any higher cut levels s in order to effectively construct good choice or ranking recommendations (see Bisdorff and Roubens, 2004).

In our example, let us suppose that the decision-maker admits the significance weights w shown in Table 1. The *Price* criterion is the most significant one with a weight of $5/15$. Then comes the *Comfort* criterion with $4/15$ and finally, both the *Speed* and the *Design* criteria have identical weights of $3/15$. By assuming that the underlying family of criteria is indeed consistent, we may state that assertion "$m_4\,\mathrm{S}_w\,m_2$" with aggregated significance of 53.3% is *rather true than false* with respect to the given importance weights w. Due to the given veto thresholds, we may also observe that both "*extreme*" models, namely m_1 – the most expensive and comfortable one, and m_7 – the cheapest and less comfortable one – appear in fact *incomparable*. A veto is triggered one way by the too important a *Price* difference: $\Delta_1(m_1, m7) = 150 \geqslant v_1$ and $r_w(m_1\,\mathrm{S}\,m_7) = 0$; and, the other way

[3] For simplicity of the exposition, we may neglect here that in principle the severeness of the veto should depend on the level of the concordance (see Roy and Bouyssou, 1993).

round, by the too important a *Comfort* difference: $\Delta_2(m_7, m_1) = -2 \leqslant -v_2$ so that equally $r_w(m_7 \,\mathrm{S}\, m_1) = 0$.

The majority concordance approach obviously requires a precise numerical measurement of the significance of the criteria, a situation which appears to be difficult to achieve in practical applications of multicriteria decision aid. Substantial efforts have been concentrated on developing analysis and methods for assessing these cardinal significance weights (see Roy and Mousseau, 1992, 1996). Following this discussion, Dias and Clímaco (2002) propose to cope with imprecise significance weights by delimiting sets of potential significance weights and enriching the proposed decision recommendations with a tolerance to achieve robust recommendations.

In this paper, we shall not contribute directly to this issue but rely on the fact that in practical applications the ordinal weighting of the significance of the criteria is generally easier to assess and more robust than any precise numerical measurement.

1.3 The ordinal concordance principle

Let us assume that instead of a given cardinal weight vector w we observe a complete pre-order π on the family of criteria F which represents the significance rank each criterion takes in the evaluation of the concordance of the global outranking relation S to be constructed on A.

In our previous car selection example, we may notice for instance that the proposed significance weights model the following pre-order: *Price* $>$ *Comfort* $>$ $\{$ *Speed*, *Design*$\}$.

A precise set of numerical weights may now be compatible or not with such a given significance ranking of the criteria.

Definition 5 (π-compatibility).
A given set $w = \{w_i | g_i \in F\}$ of significance weights is called a π-compatible if and only if:

$w_i = w_j$ for all ordered pairs (g_i, g_j) of criteria which are of the same significance with respect to π;
$w_i > w_j$ for all couples (g_i, g_j) of criteria such that criterion g_i is certainly more significant than criterion g_j in the sense of π.

We denote $W(\pi)$ the set of all π-compatible weight sets w.

Definition 6 (Ordinal concordance).
For $a, b \in A$, let "$a \,\mathrm{C}_\pi\, b$" denote the fact that "a concordantly outranks b with a significant majority for every π-compatible weight set w".

$$\big(r_w(a \,\mathrm{C}\, b) > 0.5, \; \forall w \in W(\pi) \big) \quad \Rightarrow \quad a \; \mathrm{C}_\pi \; b. \tag{6}$$

In other words, the $a\,\mathrm{C}_\pi\, b$ situation is given if for all π-compatible weight sets w, the level of concordance of assertion $a\,\mathrm{C}\,b$ outranks the level of concordance of its negation $\overline{a\,\mathrm{C}\,b}$.

Proposition 1.

$$\big(r_w(a\,\mathrm{C}\,b) \;>\; r_w(\overline{a\,\mathrm{C}\,b});\; \forall w \in W(\pi) \big) \quad \Rightarrow \quad a\,\mathrm{C}_\pi\, b. \tag{7}$$

Proof. Implication (7) results immediately from the observation that:

$$\sum_{g_j \in F} w_j \cdot r(a\,\mathrm{C}_j\, b) \;>\; \sum_{g_j \in F} w_j \cdot r(\overline{a\,\mathrm{C}_j\, b}) \quad \Leftrightarrow \quad \sum_{g_j \in F} w_j \cdot r(a\,\mathrm{C}_j\, b) \;>\; \frac{1}{2}.$$

Indeed, $\forall g_j \in F$ we observe that $r(a\,\mathrm{C}_j\, b) + r(\overline{a\,\mathrm{C}_j\, b}) = 1$. This fact implies that:

$$\sum_{g_j \in F} w_j \cdot r(a\,\mathrm{C}_j\, b) \;+\; \sum_{g_j \in F} w_j \cdot r(\overline{a\,\mathrm{C}_j\, b}) \;=\; 1.$$

Coming back to our previous car selection problem, we shall later on verify that model m_6 effectively outranks all other car models following the ordinal concordance principle. With any π-compatible set of cardinal weights, model m_6 will always concordantly outrank all these car models with a '*significant*' majority of criteria.

A constructive approach for computing such ordinal concordance statements is still needed.

2 Testing for ordinal concordance

In this section, we elaborate general conditions that must be fulfilled in order to ensure an ordinal concordance in favour of a global outranking statement.

2.1 Positive and negative significance distributions

The following implication results from the definition of the ordinal concordance principle (see Definition 6) and from Proposition 1.

$$\big(r_w(a\,\mathrm{C}\,b) - r_w(\overline{a\,\mathrm{C}\,b}) \;>\; r_w(\overline{a\,\mathrm{C}\,b}) - r_w(a\,\mathrm{C}\,b); \forall w \in W(\pi)\big) \;\Rightarrow\; a\,\mathrm{C}_\pi\, b. \tag{8}$$

The inequality on the left hand side of implication (8) gives us the operational key for implementing a test for ordinal concordance of an outranking situation. The same weights w_j and $-w_j$, denoting the "*confirming*", respectively the "*negating*", significance of each criterion, appear on each side of this inequality. Furthermore, the sum of the coefficients $r(a\,\mathrm{C}_j\, b)$ and $r(\overline{a\,\mathrm{C}_j\, b})$ – that constitute the terms $r_w(a\,\mathrm{C}\,b)$ and $r_w(\overline{a\,\mathrm{C}\,b})$ – is a constant equal to n, i.e. the number of criteria in F (see equation (2) in Definition 1). These coefficients may appear therefore as some kind of credibility distribution on the set of positive and negative significance weights.

Suppose that the given pre-order π of significance of the criteria contains k equivalence classes denoted $\pi_{(k+1)}, ..., \pi_{(2k)}$ in increasing sequence. The same equivalence classes, but in in reversed order, appearing on the "negating" significance side, are denoted $\pi_{(1)}, ..., \pi_{(k)}$.

Definition 7 (Repartition functions).
For each equivalence class $\pi_{(i)}$, we denote $w_{(i)}$ the cumulated negating, respectively confirming, significance of all equi-significant criteria gathered in this equivalence class:

$$i = 1, ..., k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} -w_j; \quad i = k+1, ..., 2k : w_{(i)} = \sum_{g_j \in \pi_{(i)}} w_j. \tag{9}$$

We denote $c_{(i)}(a, b)$ for $i = 1, ..., k$ the sum of all coefficients $r(\overline{a\,\mathrm{C}_j\,b})$ such that $g_j \in \pi_{(i)}$ and $c_{(i)}(a, b)$ for $i = k+1, ..., 2k$ the sum of all coefficients $r(a\,\mathrm{C}_j\,b)$ such that $g_j \in \pi_{(i)}$. Similarly, we denote $\overline{c_{(i)}}(a, b)$ for $i = 1, ..., k$ the sum of all coefficients $r(a\,\mathrm{C}_j\,b)$ such that $g_j \in \pi_{(i)}$ and $\overline{c_{(i)}}(a, b)$ for $i = k+1, ..., 2k$ the sum of all coefficients $r(\overline{a\,\mathrm{C}_j\,b})$ such that $g_j \in \pi_{(i)}$.

With the help of this notation, we may rewrite implication (8) as follows:
$\forall a, b \in A$:

$$\sum_{i=1}^{2k} \left(c_{(i)}(a, b) \cdot w_{(i)} \right) > \sum_{i=1}^{2k} \left(\overline{c_{(i)}}(a, b) \cdot w_{(i)} \right);\ \forall w \in W(\pi) \quad \Rightarrow \quad a\,\mathrm{C}_\pi\,b. \tag{10}$$

Coefficients $c_{(i)}(a, b)$ and $\overline{c_{(i)}}(a, b)$ represent two distributions – one the negation of the other – on an ordinal scale determined by the increasing significance $w_{(i)}$ of the equivalence classes in $\pi_{(i)}$.

2.2 Ordinal distributional dominance

We may thus test the left hand side inequality of implication (10) with the classical stochastic dominance principle originally introduced in the context of efficient portfolio selection (see Hadar and Russel, 1969; Hanoch and Levy, 1969).

We denote $C_{(i)}$, respectively $\overline{C_{(i)}}$, the increasing cumulative sums of coefficients $c_{(1)}$, $c_{(2)}$, ..., $c_{(i)}$, respectively $\overline{c_{(1)}}$, $\overline{c_{(2)}}$, ..., $\overline{c_{(i)}}$.

Lemma 1.

$$\sum_{i=1}^{2k} c_{(i)} \cdot w_{(i)} > \sum_{i=1}^{2k} \overline{c_{(i)}} \cdot w_{(i)};\forall w \in W(\pi) \;\Leftrightarrow\; \begin{cases} C_{(i)} \leqslant \overline{C_{(i)}}, i = 1, ..., 2k; \\ \exists i \in 1, ..., 2k : C_{(i)} < \overline{C_{(i)}}. \end{cases} \tag{11}$$

Proof. Demonstration of this lemma (see for instance Fishburn, 1974) goes by rewriting the right hand inequality of equivalence (10) with the help of the repartition functions $C_{(i)}$ and $\overline{C_{(i)}}$. It readily appears then that the term by term difference of the cumulative sums is conveniently oriented by the right hand conditions of equivalence (11).

This concludes the proof of our main result.

Theorem 1.
$\forall a, b \in A$, let $C_{(i)}(a,b)$ represent the increasing cumulative sums of credibility associated with a given significance ordering of the criteria:

$$a \,\mathrm{C}_\pi\, b \quad \Leftarrow \quad \begin{cases} C_{(i)}(a,b) \leqslant \overline{C_{(i)}}(a,b), i = 1, ..., 2k \textit{ and} \\ \exists i \in 1, ..., 2k : C_{(i)}(a,b) < \overline{C_{(i)}}(a,b). \end{cases} \tag{12}$$

We observe an ordinal concordant outranking situation $a \,\mathrm{C}_\pi\, b$ between actions a and b as soon as the credibility repartition on the increasing significance ordering of criteria in favour of "$a \,\mathrm{S}\, b$" is strictly below the same repartition of criteria in favour of its negation, i.e. "$\neg(a \,\mathrm{S}\, b)$".

The preceding result gives us the operational key for testing for the presence of a global outranking situation.

2.3 Testing for ordinal concordance

Let $L_3 = \{f_\pi, u, t_\pi\}$, where: f_π means *rather false than true* with any π-compatible weights w, u means *logically undetermined* and, t_π means *rather true than false* with any π-compatible weights w. For each pair of decision actions evaluated in the performance tableau, we may compute a corresponding truth-functional evaluation representing truthfulness or falseness of the presence of ordinal concordance in favour of a given outranking situation. Similar to the classic majority concordance case (see Definition 3), we here add explicitly the no veto requirement.

Definition 8 (Ordinal concordance test).
Let π be a significance ordering of the criteria. $\forall a, b \in A$, let $C_{(i)}(a,b)$ and $\overline{C_{(i)}}(a,b)$ denote the corresponding cumulative sums of increasing sums of credibility associated with relation C. We define a truth-functional evaluation of assertion $a \,\mathrm{S}\, b$, denoted $[\![a \text{ S } b]\!]_\pi$, based on relations C_π and D and, taking values in L_3 as follows:

$$[\![a \text{ S } b]\!]_\pi = \begin{cases} t_\pi & \textit{if} \quad (a \,\mathrm{C}_\pi\, b) \;\wedge\; (\overline{a \,\mathrm{D}\, b}) \;; \\ f_\pi & \textit{if} \quad (\overline{a \,\mathrm{C}_\pi\, b}) \;\vee\; (a \,\mathrm{D}\, b) \;; \\ u & \textit{otherwise}. \end{cases} \tag{13}$$

Table 2. Assessing the assertion "m_4 S m_5"

$\pi_{(i)}$	-Price	-Comfort	-Speed, Design	Speed,Design	Comfort	Price
$c_{(i)}$	0	0	1	1	1	1
$\overline{c_{(i)}}$	1	1	1	1	0	0
$C_{(i)}$	0	0	1	2	3	4
$\overline{C_{(i)}}$	1	2	3	4	4	4

Table 3. The ordinal concordance of the pairwise outranking

$[\![x \text{ S } y]\!]_\pi$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	u	u	u	u	f_π
m_2	t_π	-	t_π	f_π	u	f_π	u
m_3	t_π	t_π	-	f_π	u	f_π	u
m_4	t_π	t_π	t_π	-	t_π	t_π	f_π
m_5	t_π	t_π	t_π	t_π	-	t_π	u
m_6	t_π	t_π	t_π	t_π	t_π	-	t_π
m_7	f_π	f_π	u	t_π	t_π	f_π	-

where

$$(\overline{a \, \mathrm{C}_\pi \, b}) \quad \Leftarrow \quad \begin{cases} \big(\mathrm{C}_{(i)}(a,b) \geqslant \overline{\mathrm{C}_{(i)}}(a,b), i = 1, ..., 2k \big) \wedge \\ \big(\exists i \in 1, ..., 2k : \mathrm{C}_{(i)}(a,b) > \overline{\mathrm{C}_{(i)}}(a,b) \big). \end{cases}$$

Coming back to our simple example, we may now apply this test to the outranking situation between car models m_4 and m_5 for instance. In Table 2, we have represented the six increasing equi-significance classes we observe. From Table 1 we may compute the credibility $c_{(i)}$ (respectively $\overline{c_{(i)}}$) associated with the assertion that model m_4 outranks (respectively does not outrank) m_5, as well as the corresponding cumulative distributions $C_{(i)}$ and $\overline{C_{(i)}}$ shown in Table 2.

Applying our test, we notice that $[\![m_4 \text{ S } m_5]\!]_\pi = t_\pi$, i.e. it is true that the assertion "model m_4 outranks model m_5" will be supported by a more or less significant majority of criteria for all π-compatible sets of significance weights.

For information, we reproduce in Table 3 the complete $[\![a \, \mathrm{S} \, b]\!]_\pi$ evaluation on $A \times A$. It is worthwhile noticing that – faithful with the general concordance principle – the outranking statements "x S y" appearing with value t_π are warranted to be *more true than false* with every π-compatible significance weights. Similarly, those showing value f_π are warranted to be *more false than true* with any π-compatible significance weights. The other statements – appearing with credibility u – are to be considered *undetermined* (see Bisdorff, 2000). Truthfulness of the global outranking can in this case neither be rejected nor confirmed on the level of the ordinal concordance test.

Table 4. The cardinal majority concordance of the outranking of the car models

$r_w(x\,\mathrm{S}\,y)$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	.83	.67	.67	.67	.67	.00
m_2	.80	-	.72	.47	.67	.47	.67
m_3	.73	.73	-	.75	.67	.67	.67
m_4	.53	.53	.80	-	.80	.63	.67
m_5	.53	.73	.80	.80	-	.72	.67
m_6	.73	.73	.73	.73	.73	-	.83
m_7	.00	.00	.33	.53	.53	.60	-

As previously mentioned, model m_6 gives the unique dominant kernel, i.e. a stable and dominant subset of the $\{f_\pi, u, t_\pi\}$-valued global outranking graph. This decision action therefore represents a robust good choice decision candidate. It appears to be a *rather true than false* good choice with any possible π-compatible set of significance weights (see Bisdorff and Roubens, 2004).

Furthermore, we may observe that both "*extreme*" models, namely m_1 – the most expensive one – and m_7 – the cheapest one, appear in the unique absorbent kernel, i.e. stable and absorbent subset of the same $\{f_\pi, u, t_\pi\}$-valued global outranking graph. Thus, they constitute both bad choices with respect to with ordinal concordance requirement.

Let us now address this robustness issue directly.

3 Analyzing the robustness of global outrankings

In this last section, we are going to integrate in a same logical framework, three levels of concordance: simple majority – , ordinal – and unanimous concordance.

3.1 Three levels of concordance

Let us suppose that the decision maker has given a precise set w of significance weights with underlying significance pre-order π. The classical majority concordance approach will thus deliver an outranking index $r_w(a\,\mathrm{S}_w\,b)$ on $A \times A$.

In our car selection problem, the corresponding result is shown in Table 4. We may notice here that $r_w(m_4\,\mathrm{S}\,m_5) = 80\%$ and $[\![m_4\,\mathrm{S}\,m_5]\!]_w = t_w$. But we also know from our previous investigation that $[\![m_4\,\mathrm{S}\,m_5]\!]_\pi = t_\pi$. The outranking statement "$m_4\,\mathrm{S}\,m_5$" is thus confirmed furthermore with any π-compatible weights. Let us recall that the *Price* criterion is the most significant, followed in a second position by the *Comfort* criterion. Both the *Speed* and *Design* criteria are equally in the third position.

Going a step further, we could imagine a *perfect car model* which is the cheapest, most comfortable, very fast and superiorly designed model, called m_{top}. It is not difficult to see that this model will indeed dominate all the set A with unanimous concordance; $\forall x \in A$ and any possible weight set w, we will always have $r_w(m_{top} \,\mathrm{S}\, x) > 50\%$. The outranking statements "$m_{top} \,\mathrm{S}\, x$" are indeed warranted for any possible significance weight set and in particular also for all π-compatible ones.

Definition 9 (Unanimous concordance).
$\forall a, b \in A$ *we say that "*a unanimously outranks b*", denoted "*$a\Delta b$*", if the outranking assertion "*$a \,\mathrm{C}_j\, b$*" – semiotically restricted to each individual criterion* $g_j \in F$ *– is* rather true than false.

We capture once more the potential truthfulness of the dominance statement "$a \,\mathrm{D}\, b$" with the help of a truth-functional evaluation denoted $[\![a \,\mathrm{S}\, b]\!]_\Delta$ based on the local $r(a \,\mathrm{C}_j\, b)$ indexes and taking its values in $L_3 = \{f_\Delta, u, t_\Delta\}$, where: f_Δ means *unanimously false*, t_Δ means *unanimously true* and, u means *undetermined* as usual.

$$\forall a, b \in A : \; [\![a \,\mathrm{S}\, b]\!]_\Delta \quad = \quad \begin{cases} t_\Delta & \text{if} \quad \forall g_j \in F : \; r(a \,\mathrm{C}_j\, b) > \frac{1}{2}; \\ f_\Delta & \text{if} \quad \forall g_j \in F : \; r(a \,\mathrm{C}_j\, b) < \frac{1}{2}; \\ u & \text{otherwise.} \end{cases} \tag{14}$$

In our example, we observe a dominance situation between models m_6 and m_4. On every criterion, m_6 clearly outranks m_4 (see Table 1).

We are now going to integrate all three truth-functional evaluations, i.e. the unanimous, the ordinal and the majority concordance in a common logical framework.

3.2 Integrating unanimous, ordinal and classical majority concordance

Let w represent given numerical significance weights and π the underlying significance pre-order. We define the following ordinal sequence (increasing from falsity to truth) of logical concordance degrees: – f_Δ means *unanimous concordantly false*, – f_π means *ordinal concordantly false with any π-compatible weights*, – f_w means *majority concordantly false with weights w*, – u means *undetermined*, – t_w means *majority concordantly true with weights w*, – t_π means *ordinal concordantly true with any π-compatible weights* and, – t_Δ means *unanimous concordantly true*.

On the basis of a given performance tableau, we may thus evaluate the global outranking relation S on A as follows:

Definition 10 (Robustness denotation of the global outranking).
Let $L_7 = \{f_\Delta, f_\pi, f_w, u, t_w, t_\pi, t_\Delta\}$. $\forall a, b \in A$*, we define a truth-functional eval-*

Table 5. Robustness of the outranking on the car models

$[\![x\,\mathrm{S}\,y]\!]$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
m_1	-	t_π	t_w	t_w	t_w	t_w	f_Δ
m_2	t_π	-	t_π	f_π	t_w	f_π	t_w
m_3	t_π	t_π	-	t_w	t_w	t_w	t_w
m_4	t_π	t_π	t_π	-	t_π	t_π	f_π
m_5	t_π	t_π	t_π	t_π	-	t_π	t_w
m_6	t_π	t_π	t_π	t_Δ	t_π	-	t_π
m_7	f_Δ	f_Δ	f_w	t_π	t_π	f_π	-

uation $[\![a\,\mathrm{S}\,b]\!] \in L_7$ *as follows:*

$$[\![a\,\mathrm{S}\,b]\!] \quad = \quad \begin{cases} t_\Delta & \textit{if} \quad [\![a\,\mathrm{S}\,b]\!]_\Delta = t_\Delta\ ; \\ t_\pi & \textit{if} \quad ([\![a\,\mathrm{S}\,b]\!]_\Delta \neq t_\Delta) \wedge ([\![a\,\mathrm{S}\,b]\!]_\pi = t_\pi)\ ; \\ t_w & \textit{if} \quad ([\![a\,\mathrm{S}\,b]\!]_\pi \neq t_\pi) \wedge ([\![a\,\mathrm{S}\,b]\!]_w = t_w)\ ; \\ f_\Delta & \textit{if} \quad [\![a\,\mathrm{S}\,b]\!]_\Delta = f_\Delta\ ; \\ f_\pi & \textit{if} \quad ([\![a\,\mathrm{S}\,b]\!]_\Delta \neq f_\Delta) \wedge ([\![a\,\mathrm{S}\,b]\!]_\pi = f_\pi)\ ; \\ f_w & \textit{if} \quad ([\![a\,\mathrm{S}\,b]\!]_\pi \neq f_\pi) \wedge ([\![a\,\mathrm{S}\,b]\!]_w = f_w)\ ; \\ u & \textit{otherwise}. \end{cases} \tag{15}$$

On the seven car models, following Definition 10 we obtain the results shown in Table 5. We notice in Line m_6 that model m_6 outranks all other models at t_π level. It appears therefore to be a t_π-valued dominant kernel of this valued outranking relation. All the same, both models m_3 and m_5 appear as t_w-valued dominant kernels. And, if we apply our methodology for constructing good choices from such a L_7-valued outranking relation (see Bisdorff and Roubens, 2004) we obtain a first good choice – model m_6 – at the ordinal concordance level and, a second good choice – model m_3 – at the classic majority concordance level depending on the precisely given numerical significance weights. In terms of a best choice recommendation, it appears clearly that model m_3 represents a potential good choice depending on the precisely given significance weights, whereas model m_6 appears to be a much more robust good choice in fact independent of any given significance-order compatible weights.

Let us close our paper with the presentation of a real case study.

3.3 A robust best candidate for the EURO Best Poster Award 2004

Apart from the traditional contributed and invited presentations, the Programme Committee of the 20th European Conference on Operational Research, Rhodes 2004, invited for *discussion presentations* – a new kind of EURO K conference participation consisting in a 30 minutes presentation in front of a poster in the

Table 6. Global outranking of the posters

r_w(C)	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	.58	.24	.12	.46	.68	.34	.76	.65	**.04**	.63	.08	.28
p_2	.42	-	.34	.34	.34	.42	.42	.40	.61	**.24**	.45	.34	.26
p_3	.82	.74	-	.54	.66	.98	.86	.96	.69	**.16**	.81	.58	.46
p_4	.98	.68	.62	-	.76	.98	.82	.98	.69	**.28**	.75	.70	.54
p_5	.64	.68	.72	.48	-	**1.0**	.78	.98	.69	**.26**	.75	.52	**.0**
p_6	.54	.58	.10	.10	.34	-	.42	.86	.65	**.0**	.63	.04	**.0**
p_7	.68	.72	.32	.46	.30	.86	-	.82	.65	**.10**	.69	.50	.36
p_8	.50	.60	.16	.20	.30	.66	.40	-	.71	**.02**	.67	.16	**.0**
p_9	.43	.49	.35	.35	.41	.49	.37	.49	-	**.0**	.39	.37	.35
p_{10}	**1.0**	**.80**	**1.0**	**.84**	**1.0**	**1.0**	**.90**	**1.0**	**.71**	-	**.81**	**.88**	**.80**
p_{11}	.71	.61	.37	.29	.29	.43	.39	.59	.69	**.0**	-	.31	.43
p_{12}	.98	.66	.70	.62	.64	.96	.78	.94	.69	**.32**	.75	-	.56
p_{13}	**1.0**	.76	.70	.60	.80	.80	.70	.96	.69	**.48**	.81	.64	-

style of natural sciences conferences. In order to promote this new type of poster presentations, EURO proposed a special Best Poster Award consisting of a diploma and a prize of 1000 €. Each contributor accepted in the category of the discussion presentations was invited to submit a pdf image of his poster to a five member jury.

To evaluate the submitted poster images, the Programme Committee retained the following preference dimensions: *scientific quality* (sq), *contribution to OR theory and/or practice* (ctp), *originality* (orig) and *presentation quality* (pq) in decreasing order of importance. 13 candidates actually submitted a poster in due time and the five jury members were asked to evaluate the 13 posters on each dimension with the help of ordinal criteria functions using a scale 0 (very weak) to 10 (excellent). As usual in an ordinal context, indifference thresholds were set to zero and the preference thresholds were set equal to one ordinal level difference.

All five jury members being officially equal in importance, we considered to be in the presence of a consistent family of $5 \times 4 = 20$ criteria gathered into four equi-significance classes listed hereafter in decreasing order of importance:

$\pi_{(1)} = \{sq_1, sq_2, sq_3, sq_4, sq_5\}$,
$\pi_{(2)} = \{pct_1, pct_2, pct_3, pct_4, pct_5\}$,
$\pi_{(3)} = \{orig_1, orig_2, orig_3, orig_4, orig_5\}$,
$\pi_{(4)} = \{pq_1, pq_2, pq_3, pq_4, pq_5\}$.

The cardinal significance weights associated with these four classes of equi-significant criteria were rather arbitrarily chosen as follows: $w_{sq_i} = 4$, $w_{ctp_i} = 3$, $w_{orig_i} = 2$ and $w_{pq_i} = 1$, for $i = 1$ to 4.

We are faced with the decision problem of selecting the best – in the sense of the preference dimensions retained by the Programme Committee – out of the 13 posters on the basis of a given performance tableau. We may first computed the credibility index r_w of the concordant outranking relation C shown in Table 6 using the significance weight set w above. No veto situations being expressed by

Table 7. Robust outranking of the posters

$[\![S]\!]$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}
p_1	-	t_π	f_π	f_π	f_w	t_π	f_π	t_π	t_π	f_π	t_π	f_π	f_π
p_2	f_π	-	f_π	f_π	f_π	f_π	f_π	f_π	t_π	f_π	f_w	f_π	f_π
p_3	t_π	t_π	-	t_w	t_w	t_π	t_π	t_π	t_π	f_π	t_π	t_w	f_w
p_4	t_π	t_π	t_π	-	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	t_π
p_5	t_π	t_π	t_π	f_w	-	t_Δ	t_π	t_π	t_π	f_π	t_π	t_w	f_Δ
p_6	t_w	t_π	f_π	f_π	f_π	-	f_π	t_π	t_π	f_Δ	t_π	f_π	f_Δ
p_7	t_π	t_π	f_π	f_w	f_π	t_π	-	t_π	t_π	f_π	t_π	u	f_π
p_8	u	t_π	f_π	f_π	f_π	t_π	f_π	-	t_π	f_π	t_π	f_π	f_Δ
p_9	f_π	f_π	f_π	f_π	f_π	f_π	f_π	f_π	-	f_Δ	f_π	f_π	f_π
p_{10}	t_Δ	t_π	t_Δ	t_π	t_Δ	t_Δ	t_π	t_Δ	t_π	-	t_π	t_π	t_π
p_{11}	t_π	t_π	f_π	f_π	f_π	f_π	f_π	t_π	t_π	f_Δ	-	f_π	f_π
p_{12}	t_π	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	-	t_π
p_{13}	t_Δ	t_π	t_π	t_w	t_π	t_π	t_π	t_π	t_π	f_w	t_π	t_π	-

the members of the jury, the concordance levels $r_w(\mathrm{C})$ shown in Table 6 represent directly the credibility index $r_w(\mathrm{S})$ of the global outranking relation S.

Careful inspection of this outranking index – line by line – makes it apparent that poster p_{10} represents obviously the best candidate. It alone concordantly outranks all other posters with a comfortable weighted significance of at least 71% (see Table 6 Line p_{10}). This makes it undoubtedly the unique kernel of the global concordant outranking relation. In addition, we may notice the observed 1.0 figures that appear in Table 6. They indicate unanimous (100%) concordance situations in favour of $x \,\mathrm{S}\, y$ – so, $r_w(p_{10} \,\mathrm{S}\, p_6) = 1.0$. Similarly, .0 figures (see Line p_{13} for instance) indicate unanimous concordance *against* $x \,\mathrm{S}\, y$ – such as $r_w(p_6 \,\mathrm{S}\, p_{10}) = 0$.

The question we must ask at this point is whether this precise concordant outranking may not appear as an artifact induced by our more or less arbitrarily chosen cardinal significance weights: $\{4, 3, 2, 1\}$?

To check this issue, we computed – following our methodology – the robustness degrees $[\![S]\!]$ of the global outranking statements shown in Table 7. Inspecting Line p_{10}, we notice that the previous results become positively confirmed. Indeed, with a robustness degree of t_π, i.e. *rather true than false with any* π*-compatible weights*, poster p_{10} is confirmed in the first position. There is even evidence that p_{10} effectively dominates, i.e. unanimously outranks posters p_1, p_3, p_5, p_6 and p_8. On the other hand, looking at Column p_{10} of Table 7, we also notice that no other poster concordantly outranks p_{10}. Here, the judges again unanimously rejected the outranking of p_6 and p_{11} over p_{10}.

Selecting poster p_{10} for the EURO Best Paper Award 2004[4] appeared therefore totally independent of the choice of any precise numerical significance weights

[4] Poster p_{10} on *Political Districting via Weighted Voronoï Regions* was submitted by Federica RICCA, Bruno SIMEONE and Isabella LARI from the University of Rome “La Sapienza”. Congratulations.

compatible with the given importance ordering of the four preference dimensions retained by the Programme Committee.

Conclusion

In this paper, we have presented a formal approach for assessing truthfulness of binary outranking statements on the basis of a performance tableau involving criteria of solely ordinal significance. The concept of ordinal concordance is introduced and a formal testing procedure based on distributional dominance is developed. Thus, we solve a major practical problem concerning the precise numerical knowledge of the individual significance weights that is required by the classical majority concordance principle as implemented in the ELECTRE methods. Applicability of the concordance based aggregation of preference is extended to the case where only ordinal significance of the criteria is available. Furthermore, even if precise numerical significance is available, we provide a robustness analysis of the observed preferences by integrating unanimous, ordinal and simple majority concordance in a same logical framework.

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